Calculus 0230:
Arc Length

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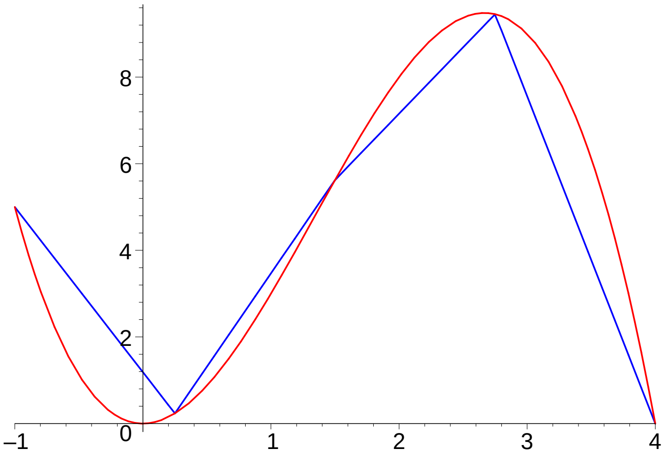
Week 6 Lecture 3

Introduction

How can we measure the length of a curve?

If the curve were a line segment then we would have no problem finding its length.

So our approach is to approximate the curve by line segments, add up their lengths, and then take the limit.



The Length of a Curve

Let C be a curve described by the parametric equations:

$$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b,$$

such that $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero for $a < t < b$ (C is **smooth**).

Approximate the length of C as follows:

- Divide $[a, b]$ into equal subintervals, each of length $\Delta t = (b - a)/n$. Let $t_0, t_1, \ldots, t_n$ be the endpoints.
- Let $x_i = f(t_i)$ and $y_i = g(t_i)$. Then $P_i = (x_i, y_i)$ is a point on C.
- Let ℓ_i be the length of the line segment from P_{i-1} to P_i. Then the total length of the approximating lines is:

  $$L_n = \sum_{i=1}^{n} \ell_i.$$

  As n increases to infinity, our approximation L_n gets ever closer to the true length, L, of the curve C:

  $$L = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} \ell_i.$$

Curve Length as an Integral

Have:

C a smooth curve, parametrized by $x = f(t)$ and $y = g(t)$ for $t \in [a, b]$.

The interval $[a, b]$ divided into subintervals of equal lengths, with endpoints $t_0, t_1, \ldots, t_n$. Let $\Delta t = (b - a)/n$.

Let $x_i = f(t_i)$, $y_i = g(t_i)$. Let $P_i = (x_i, y_i)$. Let ℓ_i be the length of the line segment from P_{i-1} to P_i.

Then the length, L, of C is: $L = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \ell_i$.

Now let $\Delta x_i = x_i - x_{i-1}$, and $\Delta y_i = y_i - y_{i-1}$. By the definition of the derivative, for Δt small,

$$\begin{array}{lcc} & f'(t_i) \approx \dfrac{\Delta x_i}{\Delta t} & g'(t_i) \approx \dfrac{\Delta y_i}{\Delta t} \\ \text{so:} & \Delta x_i \approx f'(t_i)\Delta t & \Delta y_i \approx g'(t_i)\Delta t. \end{array}$$

Hence,

$$\begin{aligned} \ell_i &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &\approx \sqrt{(f'(t_i)\Delta t)^2 + (g'(t)\Delta t)^2} \\ &= \sqrt{(f'(t))^2 + (g'(t))^2}\Delta t, \end{aligned}$$

and so,

$$L_n \approx \sum_{i=1}^{n} \sqrt{(f'(t))^2 + (g'(t))^2}\Delta t.$$

Taking the limit as $n \to \infty$ gives:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(f'(t))^2 + (g'(t))^2}\Delta t = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Arc Length Example

Theorem 1 *If a smooth curve with parametric equations* $x = f(t)$, $y = g(t)$, $t \in [a, b]$ *is traversed exactly once as* t *increases from* a *to* b, *then its length is:*

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 2 *Find the length of the arc of the curve* $x = t^3$, $y = t^2$, *that lies between the points* $(1, 1)$ *and* $(8, 4)$.

Solution. The part of the curve between $(1, 1)$ and $(8, 4)$ corresponds to the parameter interval $[1, 2]$. So the arc length formula gives:

$$\begin{aligned} L &= \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_1^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt \\ &= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt = \int_1^2 t\sqrt{4 + 9t^2}\, dy \\ &= \left[\frac{1}{27}(4 + 9t^2)^{3/2}\right]_1^2 = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13}). \end{aligned}$$

□

Non Parametric Curves

Given a curve with equation $y = f(x)$, $x \in [a, b]$, then we can regard x as the parameter. So that the parametric equations are $x = x$, $y = f(x)$, $x \in [a, b]$. Hence the Arc Length Formula becomes:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Similarly, if the curve has equation $x = g(y)$, $y \in [a, b]$ then the length is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy.$$

Evaluating the integrals which turn up in Arc Length problems is often hard and frequently impossible to do explicitly.

Non Parametric Example (I)

Example 3 *Find the length of the arc of the parabola* $y^2 = x$ *from* $(0,0)$ *to* $(1,1)$.

Solution. Since $x = y^2$, we have $dx/dy = 2y$, and hence:

$$\begin{aligned} L &= \int_0^1 \sqrt{(2y)^2+1}\,dy = \int_0^1 \sqrt{1+4y^2}\,dy \\ &= \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du \qquad \text{(substitute } u = 2y\text{)} \\ &= \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3(t)\,dt \qquad \text{(substitute } u = \tan(t)\text{)} \\ &= \frac{1}{4}\left[\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\right]_0^{\tan^{-1}(2)} \\ &= \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)\right). \end{aligned}$$

□

Non Parametric Example (II)

Example 4 *Estimate the length of the arc of the hyperbola* $xy = 1$ *from* $(1, 1)$ *to* $(2, 1/2)$.

Solution. We have,

$$y = \frac{1}{x}, \qquad \text{so} \quad \frac{dy}{dx} = -\frac{1}{x^2}, \qquad \text{over the interval} \quad [1, 2].$$

The length of the arc is then:

$$L = \int_0^1 \sqrt{1 + \frac{1}{x^4}}\, dx.$$

It is not possible to integrate this exactly. So evaluate the integral using the Midpoint Rule:

$\widehat{x}_i$	1.05	1.15	1.25	1.35	1.45	1.55	1.65	1.75	1.85	1.95
$f(\widehat{x}_i)$	1.350	1.254	1.187	1.141	1.107	1.083	1.065	1.052	1.042	1.034

Hence,

$$\begin{aligned} L \approx\ & (1.350 + 1.254 + 1.187 + 1.141 + 1.107 + 1.083 + 1.065 \\ & \qquad + 1.052 + 1.042 + 1.034)\,(0.1) \\ =\ & 1.131. \end{aligned}$$

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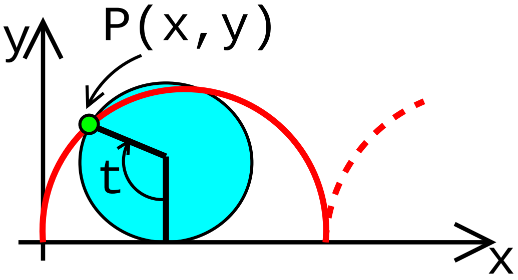
Keep on Rolling...

Consider a wheel of radius R. Fix a point P on the wheel. Now let the wheel roll on level ground and consider the path traced by P. The path is called a **cycloid**.



Let t be the angle between the vertical and the ray that extends from the center of the circle to P. Then the x and y coordinates of P are:

$$x = x(t) = R(t - \sin(t)) \qquad y = y(t) = R(1 - \cos(t)).$$

Problem 1 *Show that the length of one arch is:*

$$L = R \int_0^{2\pi} \sqrt{2(1 - \cos(t))}\, dt.$$

Use the half angle formula (below) to evaluate L:

$$\sin^2(u) = \frac{1}{2}(1 - \cos(2u)).$$

The Arc Length Function

Let C be a smooth curve with equation $y = f(x)$, $x \in [a, b]$. Let $s(x)$ be the length along C from the point $(a, f(a))$ to $(x, f(x))$.

Then s is a function on $[a, b]$, the **arc length function**, and:

$$s(x) = \int_0^x \sqrt{1 + f'(t)^2}\, dt.$$

By the Fundamental Theorem of Calculus,

$$\frac{ds}{dx} = \sqrt{1 + f'(x)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Note that the derivative of s with respect to x is always ≥ 1.

Example 5 *Find the arc length function for the curve* $y = \ln(\cos(x))$ *taking* $(0, 0)$ *for the start point.*

Solution. We have $y = f(x) = \ln(\cos(x))$. So $f'(x) = -\tan(x)$.

And hence,

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \tan^2(x)} = \sec(x).$$

So,

$$\begin{aligned} s(x) &= \int_0^x \sec(t)\, dt = [\ln|\sec(t) + \tan(t)|]_0^x \\ &= \ln|\sec(x) + \tan(x)|. \end{aligned}$$

□

Problems

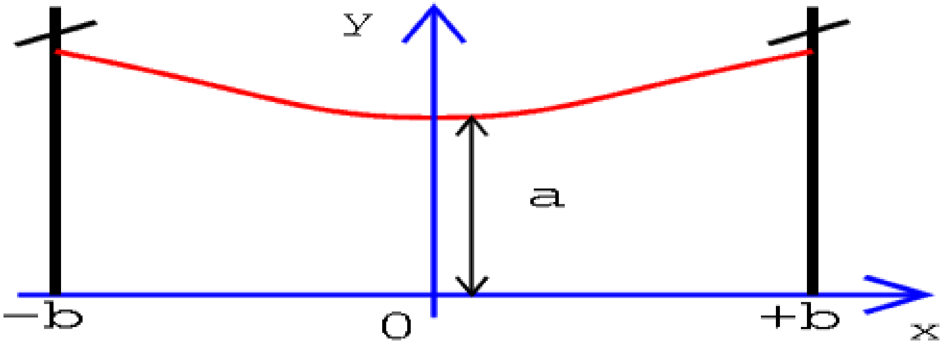
Problem 2 *Graph the curve and find its exact length:*

- $x = e^t \cos(t) \quad y = e^t \sin(t) \quad 0 \leq t \leq \pi$

Problem 3 *The figure shows a telephone wire hanging between two poles* $x = -b$ *and* $x = b$*. It takes the shape of a catenary with equation*

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right).$$

Find the length of the wire.



Hints: Note that:

$$1 + \left(\frac{1}{2}(\exp(x/a) - \exp(-x/a))\right)^2 = \left(\frac{1}{2}(\exp(x/a) + \exp(-x/a))\right)^2.$$

Problem 4

(a) *Graph the curve* $y = x^3/3 + 1/(4x)$*, with* $x > 0$.

(b) *Find the arc length function for this curve with starting point* $(1, 7/12)$.

(c) *Graph the arc length function.*

Hints: For part (b) note that:

$$1 + \left(t^2 - \frac{1}{4t^2}\right)^2 = \left(t^2 + \frac{1}{4t^2}\right)^2.$$

Solutions

The Question: Graph the curve and find its exact length:

- $x = e^t \cos(t) \quad y = e^t \sin(t) \quad 0 \leq t \leq \pi$

Solution. The derivatives of the x and y coordinate functions are:

$$x'(t) = e^t(\cos(t) - \sin(t)) \qquad y'(t) = e^t(\sin(t) + \cos(t)).$$

The Arc Length Formula for a parametric curve gives:

$$\begin{aligned} L &= \int_0^\pi \sqrt{(e^t(\cos(t) - \sin(t)))^2 + (e^t(\sin(t) + \cos(t)))^2}\, dt \\ &= \int_0^\pi \sqrt{2}e^t\, dt = \sqrt{2}(e^\pi - 1). \end{aligned}$$

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The Question: Find the length of the catenary,

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right) \qquad \text{from } -b \text{ to } b.$$

Solution. The derivative of y is,

$$y' = \frac{1}{2}\left(e^{x/a} - e^{-x/a}\right).$$

Hence the length from $-b$ to b is,

$$\begin{aligned} L &= \int_{-b}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_{-b}^{b} \sqrt{1 + \frac{1}{4}\left(e^{x/a} - e^{-x/a}\right)^2}\, dx \\ &= \int_{-b}^{b} \frac{1}{2}(e^{x/a} + e^{-x/a})\, dx \qquad \text{(using the hint)} \\ &= a\left(\exp(b/a) - \exp(-b/a)\right). \end{aligned}$$

□

The Question:

(a) Graph the curve $y = x^3/3 + 1/(4x)$, with $x > 0$.

(b) Find the arc length function for this curve with starting point $(1, 7/12)$.

(c) Graph the arc length function.

Solution. (a) and (b) — use Maple. Lets do (c).

The derivative of y is $y' = x^2 - 1/(4x^2)$. Hence the arc length function is:

$$\begin{aligned} L &= \int_1^x \sqrt{1 + \left(t^2 - \frac{1}{4t^2}\right)^2}\, dt \\ &= \int_1^x \left(t^2 + \frac{1}{4t^2}\right) dt \qquad \text{(using the hint)} \\ &= \frac{4x^4 - 3 - x}{12x}. \end{aligned}$$

□

Summary

You know:

- How the **length of a curve** is defined and interpreted as an integral
- Arc Length Formulae:
  - Parametric version

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

  - Function in x

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

  - Function in y

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy.$$

- The definition of the **arc length function**

$$s(x) = \int_0^x \sqrt{1 + f'(t)^2}\, dt.$$